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Substitutable Choice Functions and Convex Geometry

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Abstract

For a finite nonempty set E we consider a choice function $C : 2^E \to 2^E$ (i.e., $C(X) \subseteq X$ $(X \subseteq E)$) that satisfies
(A1) For any X with $\emptyset \neq X \subseteq E$, $C(X) \neq \emptyset$.
(A2) (Substitutability) For any $X \subseteq E$ and $e \in C(X)$,
$$C(X) \setminus \{e\} \subseteq C(X \setminus \{e\}).$$
We call an ordering (or permutation) $(e_1, e_2, \cdots, e_n)$ of E *admissible* if for each $i = 1, 2, \cdots, n$ we have $e_i \in C(E \setminus \{e_1, e_2, \cdots, e_{i-1}\})$.

We show that the collection of all the admissible orderings is an antimatroid. Equivalently, defining $\mathcal{F} = \{E \setminus \{e_1, e_2, \cdots, e_{i-1}\} \mid (e_1, e_2, \cdots, e_n)$: admissible ordering, $i \in \{1, 2, \cdots, n\}\}$, we get a convex geometry $(E, \mathcal{F})$. The present result reveals that a convex geometry (or an antimatroid) naturally arises from any substitutable choice function (satisfying (A1)).

Keywords:
Choice function, substitutability, convex geometry, antimatroid

1. Introduction

Koshevoy [3] and Johnson and Dean [4] pointed out the correspondence between path-independent choice functions ([6]) and convex geometries ([2]) (also see [1, 5]).

In the present note we consider choice functions that satisfy a substitutability condition, which is weaker than the path-independence condition. We will show that every substitutable choice function yields a convex geometry, which reveals a lattice structure and a closure space behind any substitutable choice function through the associated convex geometry.

2. Definitions and Assumptions

Let E be a finite set with its cardinality $|E| = n$. We consider a choice function $C : 2^E \to 2^E$, i.e.,

$$\forall X \subseteq E : C(X) \subseteq X. \tag{1}$$

We assume

(A1) For any X with $\emptyset \neq X \subseteq E$ we have $C(X) \neq \emptyset$.

An ordering (or permutation) $(e_1, e_2, \cdots, e_n)$ of E is called *admissible* if for each $i = 1, 2, \cdots, n$

$$e_i \in C(E \setminus \{e_1, e_2, \cdots, e_{i-1}\}). \tag{2}$$

Here, note that at least one admissible ordering of E exists due to Assumption (A1). (Imagine the repeating process of choosing an element from the set specified by the choice function for a current underlying set (initially E) and removing the chosen element from the current underlying set.) We call each initial segment $(e_1, e_2, \cdots, e_i)$ $(i = 1, 2, \cdots, n)$ of an admissible ordering $(e_1, e_2, \cdots, e_n)$ of E an *admissible sequence* and a set $\{e_1, e_2, \cdots, e_i\}$ an *admissible set*. Let $\mathcal{F}$ be the collection of complements of admissible sets, i.e.,

$$\mathcal{F} = \{E \backslash \{e_1, e_2, \cdots, e_{i-1}\} \mid (e_1, e_2, \cdots, e_n) : \text{ admissible ordering}, i \in \{1, 2, \cdots, n\}\} \tag{3}$$

We also assume

(A2) For any $X \in \mathcal{F}$ and $e \in C(X)$ we have

$$C(X) \setminus \{e\} \subseteq C(X \setminus \{e\}). \tag{4}$$

This is a *substitutability* condition for choice function C.

3. An Associated Convex Geometry

We show that the collection $\mathcal{F}$ given by (3) provides us with a convex geometry $(E, \mathcal{F})$ on E.

We first show the following basic lemma.

Lemma 3.1. *For any admissible ordering* $(e_1, e_2, \cdots, e_n)$*, if we have*

$$e_j \in C(E \setminus \{e_1, e_2, \cdots, e_{i-1}\}) \tag{5}$$

for some integers i *and* j *with* $1 \leq i < j \leq n$*, then for any integer* k *with* $i \leq k \leq j-1$ *a new ordering given by*

$$(e_1, \cdots, e_{k-1}, e_j, e_k, \cdots, e_{j-1}, e_{j+1}, \cdots, e_n) \tag{6}$$

is also admissible. (The new ordering is obtained by shifting e_j *to the position immediately before* e_k*.)*
(Proof) For any integer l with $i \leq l \leq j-1$ define $X_l = E \setminus \{e_1, e_2, \cdots, e_{l-1}\}$. Because of the assumption we have $e_i, e_j \in C(X_i)$. Hence it follows from Assumption (A2) that for any given k with $i \leq k \leq j-1$

$$e_l, e_j \in C(X_l) \qquad (k \leq l \leq j-1), \tag{7}$$

which implies, due to (A2) again,

$$e_l \in C(X_l) \setminus \{e_j\} \subseteq C(X_l \setminus \{e_j\}) \qquad (k \leq l \leq j-1). \tag{8}$$

Hence the ordering given by (6) is admissible. □

From this lemma we show the following.

Lemma 3.2. *For any distinct* $X, Y \in \mathcal{F}$ *we have* $X \cap Y \in \mathcal{F}$*.*
(Proof) Suppose that $X = E \setminus \{e_1, e_2, \cdots, e_k\}$ and $Y = E \setminus \{e'_1, e'_2, \cdots, e'_l\}$ for admissible sequences $L_X = (e_1, e_2, \cdots, e_k)$ and $L_Y = (e'_1, e'_2, \cdots, e'_l)$. Also suppose that

$$e_1 = e'_1, \ \cdots, \ e_p = e'_p, \ e_{p+1} \neq e'_{p+1} \tag{9}$$

for some integer p with $0 \leq p < \min\{k, l\}$.

Now there exist the following three cases I∼III.
[Case I: $e'_{p+1} \notin X$]

Since $e'_{p+1} = e_q$ for some q with $p+1 < q \leq k$, from Lemma 3.1,

$$(e_1, \cdots, e_p, e'_{p+1}, e_{p+1}, \cdots, e_{q-1}, e_{q+1}, \cdots, e_k) \tag{10}$$

is also an admissible sequence that gives X. Replacing L_X by sequence (10), we get a new admissible sequence L_X that has a longer common initial segment with L_Y.

[Case II: $e_{p+1} \notin Y$]

Similarly as in Case I, we can get a new admissible sequence L_Y that has a longer common initial segment with L_X.

[Case III: $e'_{p+1} \in X \setminus Y$ and $e_{p+1} \in Y \setminus X$]

In the present case, due to Lemma 3.1 we get new admissible sequences (longer by one)

$$(e_1, \cdots, e_p, e'_{p+1}, e_{p+1}, \cdots, e_k), \tag{11}$$

$$(e'_1, \cdots, e'_p, e'_{p+1}, e_{p+1}, e'_{p+2}, \cdots, e'_l), \tag{12}$$

the length of whose common initial segment becomes larger by two than that of L_X and L_Y. Also note that they give admissible sets $E \setminus (X \setminus \{e'_{p+1}\})$ and $E \setminus (Y \setminus \{e_{p+1}\})$. So we replace X and Y by $X \setminus \{e'_{p+1}\}$ and $Y \setminus \{e_{p+1}\}$, respectively, and new L_X and L_Y are given by (11) and (12), respectively.

While there exists p such that (9) holds, update X, Y, L_X, L_Y as shown in Cases I$\sim$III described above. We eventually obtain X and Y such that (i) $E \setminus X$ and $E \setminus Y$ are admissible sets and (ii) $X \subseteq Y$ or $Y \subseteq X$, and then X or Y is equal to the original $X \cap Y$. Hence the original $X \cap Y$ satisfies $X \cap Y \in \mathcal{F}$. □

From this we obtain our main result as follows.

Theorem 3.3. *$(E, \mathcal{F})$ is a convex geometry.*

(Proof) Since $\emptyset, E \in \mathcal{F}$ and for any nonempty $X \in \mathcal{F}$ there exists some $e \in X$ such that $X \setminus \{e\} \in \mathcal{F}$ due to (A1), it follows from Lemma 3.2 that $(E, \mathcal{F})$ is a convex geometry. □

4. Concluding Remarks

We have shown that a convex geometry arises from any substitutable choice function. It should be noted that our result depends only on Assumptions (A1) and (A2). Hence $C(X)$ $(X \in 2^E \setminus \mathcal{F})$ do not affect the structure of the associated convex geometry. It should also be noted that defining a closure operator cl by $\mathrm{cl}(X) = \cap\{Y \mid X \subseteq Y \in \mathcal{F}\}$ $(X \subseteq E)$ and a new choice function $\hat{C}(X) = C(\mathrm{cl}(X))$ $(X \subseteq E)$, we get a path-independent choice function $\hat{C}$ that gives the same associated convex geometry $(E, \mathcal{F})$.

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